C^*-Correspondences for Ordinal Graphs

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Directed graphs

Definition

A directed graph is a collection $E = \left(E^0, E^1, r, s\right)$ for which

- E^0 is the set of vertices
- E^1 is the set of edges
- $r, s : E^1 \to E^0$ are the range and source functions

The C^*-algebra $C^*(E)$ is universal for mutually orthogonal projections $\left\{T_v : v \in E^0\right\}$ and partial isometries $\left\{T_e : e \in E^1\right\}$ satisfying the following relations:

1. $T_e^* T_e = T_{s(e)}$ for all $e \in E^1$
2. $T_{r(e)} T_e = T_e$ for all $e \in E^1$
3. $T_e^* T_f = 0$ for all distinct $e, f \in E^1$
4. $\sum_{e \in r^{-1}(v)} T_e T_e^* = T_v$ for all $v \in E^0$ satisfying $0 < \left|r^{-1}(v)\right| < \infty$

A vertex v satisfying $0 < \left|r^{-1}(v)\right| < \infty$ is said to be regular.

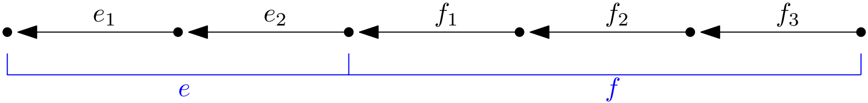
Composition of paths

Definition

A path e is a finite collection of edges $e_1, \ldots e_n$ such that $s(e_k) = r(e_{k+1})$. Define the length of e to be $d(e) = n$.

Can regard a vertex as a path of length 0.

Can compose paths e and f to get ef as follows:



Note: The edges in the paths e and f are well-ordered. The edges in ef are well-ordered, with order-isomorphism class coming from the *ordinal* sum $d(e) + d(f)$.

Composition of longer paths

What if we allow $d\,(e)$ and $d\,(f)$ to take values in Ord outside of $[0, \omega)$?

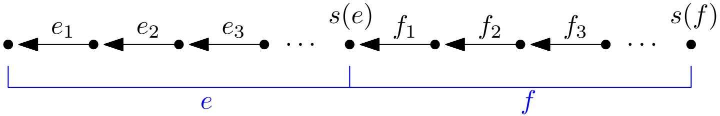


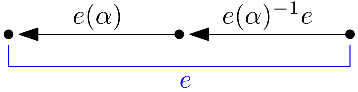
Figure: Composition of paths e, f with $d\,(e) = d\,(f) = \omega$ and $d\,(ef) = d\,(e) + d\,(f) = \omega \cdot 2$

Ordinal graphs

Definition

An ordinal graph is a small category Λ with a functor $d : \Lambda \to \mathrm{Ord}$ satisfying the following factorization property:

> For every morphism $e \in \Lambda$ and $\alpha \leq d(e)$ there exist unique morphisms $e(\alpha), e(\alpha)^{-1} e \in \Lambda$ such that $d(e(\alpha)) = \alpha$ and $e = e(\alpha)\, e(\alpha)^{-1} e$



Ordinal graph C^*-algebras

Every ordinal graph is automatically left-cancellative since ordinal addition is left-cancellative, so it has a C^*-algebra defined by Spielberg which I will denote $C^*(\Lambda)$.

$C^*(\Lambda)$ is universal for generators and relations:

$$\{T_v : d(v) = 0\} \sqcup \{T_e : d(e) = \omega^\alpha \text{ for some } \alpha \in \mathrm{Ord}\}$$

1. $T_e^* T_e = T_{s(e)}$
2. $T_e T_f = T_{ef}$ if $s(e) = r(f)$ and $d(e) < d(f)$
3. $T_e^* T_f = 0$ if $e\Lambda \cap f\Lambda = \emptyset$
4. $T_v = \sum_{e \in \Lambda^{\omega^\alpha}} T_e T_e^*$ if v is an α-regular vertex

α-regular vertices

Convenient notation:

$$\Lambda_\alpha = \{e \in \Lambda : d(e) < \omega^\alpha\}$$

$$\Lambda^\alpha = \{e \in \Lambda : d(e) = \alpha\}$$

Definition

A vertex $v \in \Lambda_0$ is *α-source-regular* if for every $e \in v\Lambda_\alpha$, $s(e)\Lambda^{\omega^\alpha} \neq \emptyset$. Define v to be *α-row-finite* if $v\Lambda^{\omega^\alpha}$ is finite, and v to be *α-regular* if v is α-source-regular and α-row-finite.

Examples

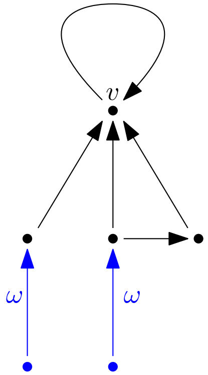


Figure: v is 1-source-regular, but not 1-row-finite.

Examples

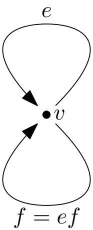


Figure: v is 1-regular

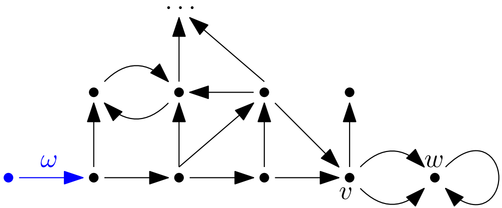


Figure: v is 1-regular but w is not

C^*-correspondences

Definition

For each $\alpha \in \mathrm{Ord}$, define X_α to be the completion of

$$\left\{ f \in c_c\left(\Lambda^{\omega^\alpha}, C^*\left(\Lambda_\alpha\right)\right) : T_{s(e)} f\left(e\right) = f\left(e\right) \text{ for all } e \in \Lambda^{\omega^\alpha} \right\}$$

with the following operations which make it a C^*-correspondence:

$$\left(x \cdot a\right)\left(e\right) = x\left(e\right) a$$

$$\langle x, y \rangle = \sum_{e \in \Lambda^{\omega^\alpha}} x\left(e\right)^* y\left(e\right)$$

$$\left(T_g^* \cdot x\right)\left(e\right) = \begin{cases} x\left(ge\right) & s\left(g\right) = r\left(e\right) \\ 0 & \text{otherwise} \end{cases}$$

Results

Theorem

Suppose for each $\alpha \in \mathrm{Ord}$ *and path* $f \in \Lambda^{\omega^\alpha}$ *such that* $r(f)$ *is* α*-regular,* $ef = f$ *implies* e *is a vertex. Then we have the following:*

1. *For every* $\alpha \in \mathrm{Ord}$*, the homomorphisms* $\rho_\alpha : C^*(\Lambda_\alpha) \to C^*(\Lambda)$ *defined by*
$$\rho_\alpha(S_e) = T_e$$
are injective.
2. *For every* $\alpha \in \mathrm{Ord}$*, the representation* $(\psi_\alpha, \rho_\alpha) : (X_\alpha, C^*(\Lambda_\alpha)) \to C^*(\Lambda_{\alpha+1})$ *of* X_α *defined by*
$$\psi_\alpha(\delta_e) = T_e$$
induces an isomorphism $\psi_\alpha \times \rho_\alpha : \mathcal{O}_{X_\alpha} \to C^*(\Lambda_{\alpha+1})$*.*

Results

Corollary

Suppose Λ *satisfies condition (S). Then we have the following:*

1. *For every* $\alpha \in \mathrm{Ord}$, X_α *satisfies condition (S).*
2. *A* **-homomorphism* $\pi : C^*(\Lambda) \to A$ *into a* C^**-algebra* A *is injective iff* $\pi(T_v) \neq 0$ *for every vertex* $v \in \Lambda_0$.