# $C^*$-correspondences for Ordinal Graphs

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# Ordinal Graph $C^*$-algebras

## Definition

An ordinal graph is a small category $\Lambda$ with a functor $d : \Lambda \to \mathrm{Ord}$ satisfying the following factorization property:

> For every morphism $e \in \Lambda$ and $\alpha \leq d(e)$ there exist unique morphisms $e(\alpha), e(\alpha)^{-1} e \in \Lambda$ such that $d(e(\alpha)) = \alpha$ and $e = e(\alpha)\, e(\alpha)^{-1} e$

Each ordinal graph $\Lambda$ is left-cancellative, and hence has a $C^*$-algebra $C^*(\Lambda)$ universal for generators and relations

$$\{T_v : d(v) = 0\} \sqcup \{T_e : d(e) = \omega^\alpha \text{ for some } \alpha \in \mathrm{Ord}\}$$

1. $T_e^* T_e = T_{s(e)}$
2. $T_e T_f = T_{ef}$ if $s(e) = r(f)$ and $d(e) < d(f)$
3. $T_e^* T_v = 0$ if $e\Lambda \cap f\Lambda = \emptyset$
4. $T_v = \sum_{e \in \Lambda^{\omega^\alpha}} T_e T_e^*$ if $v$ is an $\alpha$-regular vertex

# $C^*$-correspondences

## Definition

Define $\Lambda_\alpha = \{e \in \Lambda : d(e) < \omega^\alpha\}$, $\Lambda^\alpha = \{e \in \Lambda : d(e) = \alpha\}$, and

$$X_\alpha = \left\{f \in c_c\left(\Lambda^{\omega^\alpha}, C^*(\Lambda_\alpha)\right) : T_{s(e)} f(e) = f(e) \text{ for all } e \in \Lambda^{\omega^\alpha}\right\}$$

Then each $X_\alpha$ is a $C^*$-correspondence with operations

$$(x \cdot T_g)(e) = x(e)\, T_g$$

$$\left(T_g^* \cdot x\right)(e) = \begin{cases} x(ge) & s(g) = r(e) \\ 0 & \text{otherwise} \end{cases}$$

$$\langle x, y \rangle = \sum_{e \in \Lambda^{\omega^\alpha}} x(e)^* y(e)$$

# Result

## Theorem

Let $\Lambda$ be an ordinal graph such that for each $\alpha \in \mathrm{Ord}$ and $f \in \Lambda^{\omega^\alpha}$ with $r(f)$ $\alpha$-regular, $ef = f$ implies $d(e) = 0$. Then

1. The homomorphisms $\rho_\alpha : C^*(\Lambda_\alpha) \to C^*(\Lambda)$ defined by $\rho_\alpha(S_e) = T_e$ are all injective.
2. The Katsura ideal $J_\alpha$ for $X_\alpha$ is the ideal in $C^*(\Lambda_\alpha)$ generated by $\{S_v : v \text{ is an } \alpha\text{-regular vertex}\}$.
3. $\left(\psi_\alpha, \rho_\alpha^{\alpha+1}\right) : (X_\alpha, C^*(\Lambda_\alpha)) \to C^*(\Lambda_{\alpha+1})$ defined by
$$\psi_\alpha(\delta_f) = T_f$$
$$\rho_\alpha^{\alpha+1}(S_e) = T_e$$
is a covariant representation of $X_\alpha$.
4. The induced homomorphisms $\psi_\alpha \times \rho_\alpha^{\alpha+1} : \mathcal{O}_{X_\alpha} \to C^*(\Lambda_{\alpha+1})$ are isomorphisms.